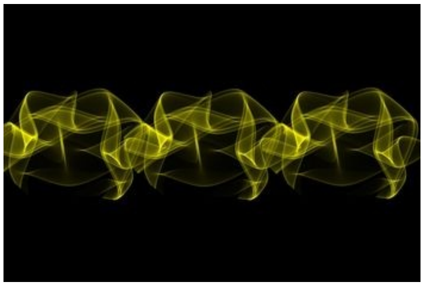
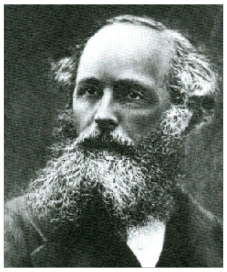
Introduction to Electromagnetic Theory

Lecture topics

- Laws of magnetism and electricity
- Meaning of Maxwell’s equations
- Solution of Maxwell’s equations



Electromagnetic radiation: wave model



• James Clerk Maxwell (1831-1879) – Scottish mathematician and physicist

• Wave model of EM energy

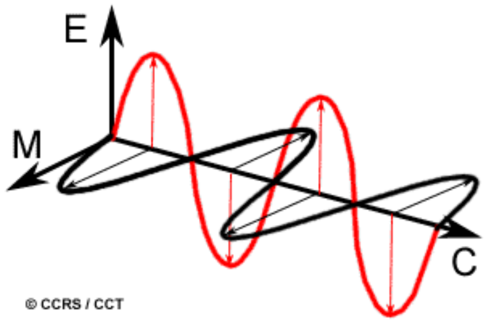
  • Unified existing laws of electricity and magnetism (Newton, Faraday, Kelvin, Ampère)

  • Oscillating electric field produces a magnetic field (and vice versa) – propagates an EM wave

  • Can be described by 4 differential equations

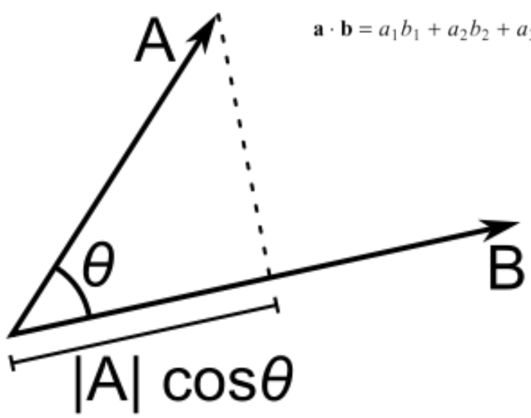
  • Derived speed of EM wave in a vacuum

Electromagnetic radiation



- EM wave is:
- Electric field (E) perpendicular to magnetic field (M)
- Travels at velocity, c ($3x10^8$ ms^{-1}, in a vacuum)

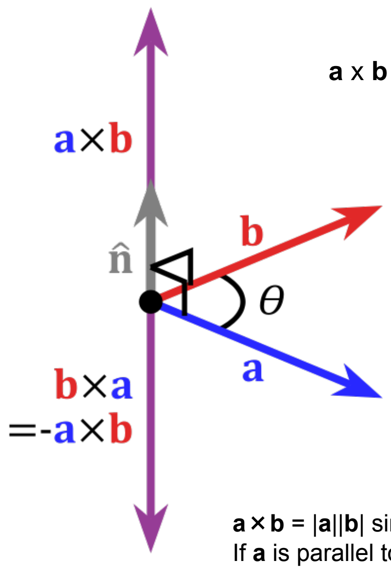
Dot (scalar) product



$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 = |\mathbf{a}||\mathbf{b}| \cos\theta$$

A•B = |**A**||**B**|cos θ
If **A** is perpendicular to **B**, the dot product of **A** and **B** is zero

Cross (vector) product



a x **b** = $[(a_2b_3 - a_3b_2), (a_3b_1 - a_1b_3), (a_1b_2 - a_2b_1)]$

a × **b** = |**a**||**b**| sin θ **n**

If **a** is parallel to **b**, the cross product of **a** and **b** is zero

Div, Grad, Curl

Types of 3D vector derivatives:

The Del operator:

$$\vec{\nabla} \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$

The Gradient of a scalar function *f* (vector):

$$\vec{\nabla} f \equiv \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

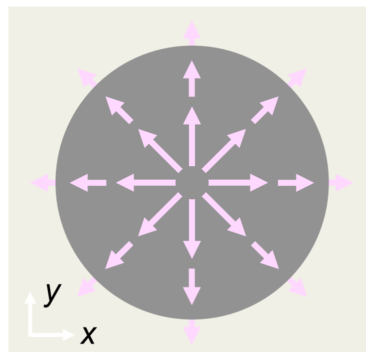
Div, Grad, Curl

The Divergence of a vector function (scalar):

$$\nabla \cdot f = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z}$$

The Divergence is nonzero if there are sources or sinks.

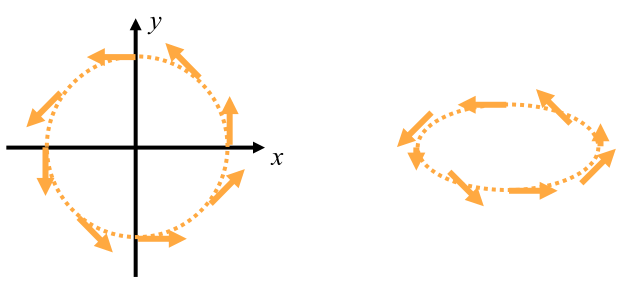
A 2D source with a large divergence:



Div, Grad, Curl

The Curl of a vector function $\vec{f}$:

$$\vec{\nabla} \times \vec{f} \equiv \left(\frac{\partial f_z}{\partial y} - \frac{\partial f_y}{dz}, \frac{\partial f_x}{\partial z} - \frac{\partial f_z}{dx}, \frac{\partial f_y}{\partial x} - \frac{\partial f_x}{dy} \right)$$



Functions that tend to **curl around** have large curls.

http://mathinsight.org/curl_idea

Div, Grad, Curl

The Laplacian of a scalar function :

$$\nabla^2 f \equiv \vec{\nabla}\cdot\vec{\nabla} f = \vec{\nabla}\cdot\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

The Laplacian of a vector function is the same, but for each component of *f*:

$$\nabla^2 \vec{f} = \left(\frac{\partial^2 f_x}{\partial x^2} + \frac{\partial^2 f_x}{\partial y^2} + \frac{\partial^2 f_x}{\partial z^2}, \frac{\partial^2 f_y}{\partial x^2} + \frac{\partial^2 f_y}{\partial y^2} + \frac{\partial^2 f_y}{\partial z^2}, \frac{\partial^2 f_z}{\partial x^2} + \frac{\partial^2 f_z}{\partial y^2} + \frac{\partial^2 f_z}{\partial z^2}\right)$$

The Laplacian tells us the curvature of a vector function.

Maxwell's Equations

- Four equations relating electric (**E**) and magnetic fields (**B**) – vector fields
- ε_0 is electric permittivity of free space (or vacuum permittivity - a constant) – *resistance to formation of an electric field in a vacuum*
- $\varepsilon_0 = 8.854188\times10^{-12}$ Farad m^{-1}
- μ_0 is magnetic permeability of free space (or magnetic constant - a constant) – *resistance to formation of a magnetic field in a vacuum*
- $\mu_0 = 1.2566\text{x}10^{-6}$ T.m/A (T = Tesla; SI unit of magnetic field)

$$\nabla \bullet E = \frac{\rho}{\varepsilon_0}$$

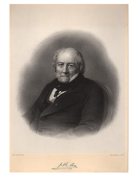
$$\nabla \bullet B = 0$$

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

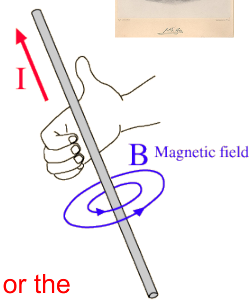
$$\nabla \times B = \mu_0 J + \varepsilon_0 \mu_0 \frac{\partial E}{\partial t}$$

Note: $\nabla\bullet$ is 'divergence' operator and ∇x is 'curl' operator

Biot-Savart Law (1820)



- Jean-Baptiste Biot and Felix Savart (French physicist and chemist)
- The magnetic field **B** at a point a distance **R** from an infinitely long wire carrying current **I** has magnitude:



$$B = \frac{\mu_0 I}{2\pi R}$$

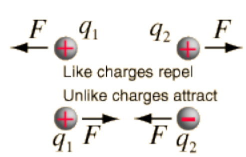
- Where μ_0 is the magnetic permeability of free space or the magnetic constant
- Constant of proportionality linking magnetic field and distance from a current
- Magnetic field strength decreases with distance from the wire
- $\mu_0 = 1.2566\text{x}10^{-6}$ T.m/A (T = Tesla; SI unit of magnetic field)

Coulomb's Law (1783)

- Charles Augustin de Coulomb (French physicist)
- The magnitude of the electrostatic force (F) between two point electric charges (q_1, q_2) is given by:

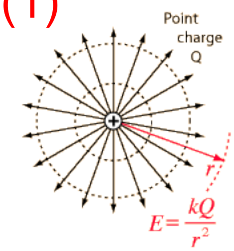
$$F = \frac{q_1 q_2}{4\pi\varepsilon_0 r^2}$$

- Where ε_0 is the electric permittivity or electric constant
- Like charges repel, opposite charges attract
- $\varepsilon_0 = 8.854188 \times 10^{-12}$ Farad m^{-1}

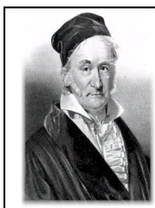


Maxwell's Equations (1)

$$\nabla \bullet E = \frac{\rho}{\varepsilon_0}$$

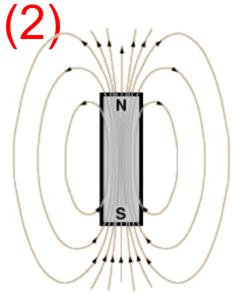


- Gauss' law for electricity: the electric flux out of any closed surface is proportional to the total charge enclosed within the surface; i.e. a charge will radiate a measurable field of influence around it.
- **E** = electric field, ρ = net charge inside, ε_0 = vacuum permittivity (constant)
- Recall: divergence of a vector field is a measure of its tendency to converge on or repel from a point.
- Direction of an electric field is the direction of the force it would exert on a positive charge placed in the field
- If a region of space has more electrons than protons, the total charge is negative, and the direction of the electric field is negative (inwards), and vice versa.

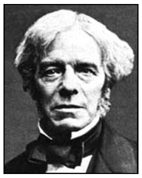


Maxwell's Equations (2)

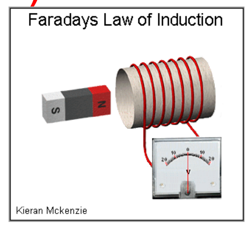
$$\nabla \bullet B = 0$$



- Gauss' law for magnetism: the net magnetic flux out of any closed surface is zero (i.e. magnetic monopoles do not exist)
- **B** = magnetic field; magnetic flux = **B**A (A = area perpendicular to field **B**)
- Recall: divergence of a vector field is a measure of its tendency to converge on or repel from a point.
- Magnetic sources are dipole sources and magnetic field lines are loops – we cannot isolate N or S 'monopoles' (unlike electric sources or point charges – protons, electrons)
- Magnetic monopoles *could* exist, but have never been observed



Maxwell's Equations (3)



$$\nabla \times E = -\frac{\partial B}{\partial t}$$

- Faraday's Law of Induction: the curl of the electric field (**E**) is equal to the negative of rate of change of the magnetic flux through the area enclosed by the loop
- **E** = electric field; **B** = magnetic field
- Recall: curl of a vector field is a vector with magnitude equal to the maximum 'circulation' at each point and oriented perpendicularly to this plane of circulation for each point.
- Magnetic field weakens ➔ curl of electric field is positive and vice versa
- Hence changing magnetic fields affect the curl ('circulation') of the electric field – basis of electric generators (moving magnet induces current in a conducting loop)

Maxwell's Equations (4)

$$\nabla \times B = \mu_0 J + \varepsilon_0 \mu_0 \frac{\partial E}{\partial t}$$

- Ampère's Law: the curl of the magnetic field (**B**) is proportional to the electric current flowing through the loop

AND to the rate of change of the electric field. 🡐 added by Maxwell

- **B** = magnetic field; J = current density (current per unit area); **E** = electric field
- The curl of a magnetic field is basically a measure of its strength
- First term on RHS: in the presence of an electric current (J), there is always a magnetic field around it; **B** is dependent on J (*e.g., electromagnets*)
- Second term on RHS: a changing electric field generates a magnetic field.
- Therefore, generation of a magnetic field does not require electric current, only a changing electric field. An oscillating electric field produces a variable magnetic field (as dE/dT changes)

Putting it all together….

- An oscillating electric field produces a variable magnetic field. A changing magnetic field produces an electric field….and so on.
- In ‘free space’ (vacuum) we can assume current density (J) and charge (ρ) are zero i.e. there are no electric currents or charges
- Equations become:

$$\nabla \bullet E = 0$$

$$\nabla \bullet B = 0$$

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

$$\nabla \times B = \varepsilon_0 \mu_0 \frac{\partial E}{\partial t}$$

Solving Maxwell’s Equations

Take curl of: $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

$$\vec{\nabla} \times [\vec{\nabla} \times \vec{E}] = \vec{\nabla} \times [-\frac{\partial \vec{B}}{\partial t}]$$

Change the order of differentiation on the RHS:

$$\vec{\nabla} \times [\vec{\nabla} \times \vec{E}] = -\frac{\partial}{\partial t}[\vec{\nabla} \times \vec{B}]$$

Solving Maxwell's Equations (cont'd)

But (Equation 4): $$\vec{\nabla} \times \vec{B} = \mu\varepsilon \frac{\partial \vec{E}}{\partial t}$$

Substituting for $\vec{\nabla} \times \vec{B}$, we have:

$$\vec{\nabla} \times [\vec{\nabla} \times \vec{E}] = -\frac{\partial}{\partial t}[\vec{\nabla} \times \vec{B}] \Rightarrow \vec{\nabla} \times [\vec{\nabla} \times \vec{E}] = -\frac{\partial}{\partial t}[\mu\varepsilon \frac{\partial \vec{E}}{\partial t}]$$

Or:

$$\vec{\nabla} \times [\vec{\nabla} \times \vec{E}] = -\mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

assuming that μ and ε are constant in time.

Solving Maxwell's Equations (cont'd)

Identity: $$\vec{\nabla} \times [\vec{\nabla} \times \vec{f}] \equiv \vec{\nabla}(\vec{\nabla} \cdot \vec{f}) - \nabla^2 \vec{f}$$

Using the identity, $\vec{\nabla} \times [\vec{\nabla} \times \vec{E}] = -\mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}$

becomes: $\vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E} = -\mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}$

Assuming zero charge density (free space; Equation 1):

$$\vec{\nabla} \cdot \vec{E} = 0$$

and we're left with: $$\nabla^2 \vec{E} = \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

Solving Maxwell's Equations (cont'd)

$$\nabla^2 \vec{E} = \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} \qquad \nabla^2 \vec{B} = \mu\varepsilon \frac{\partial^2 \vec{B}}{\partial t^2}$$

The same result is obtained for the magnetic field B.

These are forms of the 3D wave equation, describing the propagation of a sinusoidal wave:

$$\nabla^2 u = \frac{1}{v^2} \frac{\partial^2 u}{\partial t^2}$$

Where v is a constant equal to the propagation speed of the wave

So for EM waves, v = $\frac{1}{\sqrt{\mu\varepsilon}}$

Solving Maxwell's Equations (cont'd)

So for EM waves, v = $\frac{1}{\sqrt{\mu\varepsilon}}$,

Units of μ = T.m/A

The Tesla (T) can be written as kg A^{-1} s^{-2}

So units of μ are kg m A^{-2} s^{-2}

Units of ε = Farad m^{-1} or A^2 s^4 kg^{-1} m^{-3} in SI base units

So units of με are m^{-2} s^2

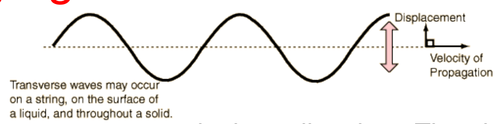
Square root is m^{-1} s, reciprocal is m s^{-1} (i.e., velocity)

$\varepsilon_0 = 8.854188 \times 10^{-12}$ and $\mu_0 = 1.2566371 \times 10^{-6}$

Evaluating the expression gives 2.998×10^8 m s^{-1}

Maxwell (1865) recognized this as the (known) speed of light – confirming that light was in fact an EM wave.

Why light waves are transverse



Suppose a wave propagates in the *x*-direction. Then it's a function of *x* and *t* (and not *y* or *z*), so all *y*- and *z*-derivatives are zero:

$$\frac{\partial E_y}{\partial y} = \frac{\partial E_z}{\partial z} = \frac{\partial B_y}{\partial y} = \frac{\partial B_z}{\partial z} = 0$$

In a charge-free medium,

$$\vec{\nabla} \cdot \vec{E} = 0 \;\; and \;\; \vec{\nabla} \cdot \vec{B} = 0$$

that is,

$$\frac{\partial E_x}{\partial x} + \cancel{\frac{\partial E_y}{\partial y}} + \cancel{\frac{\partial E_z}{\partial z}} = 0 \qquad \frac{\partial B_x}{\partial x} + \cancel{\frac{\partial B_y}{\partial y}} + \cancel{\frac{\partial B_z}{\partial z}} = 0$$

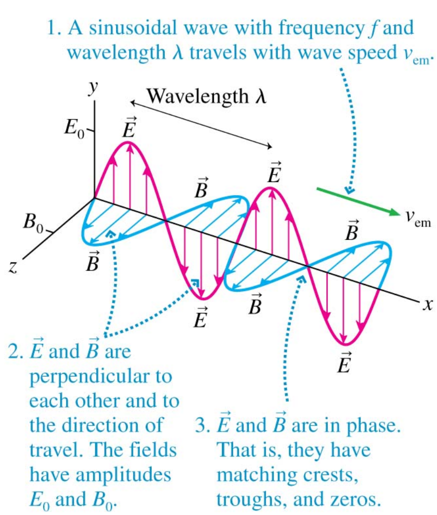
Substituting the zero values, we have:

$$\frac{\partial E_x}{\partial x} = 0 \;\text{ and }\; \frac{\partial B_x}{\partial x} = 0$$

So the longitudinal fields (parallel to propagation direction) are at most **constant**, and not waves.

The propagation direction of a light wave

FIGURE 35.19 A sinusoidal electromagnetic wave.



$$\vec{v} = \vec{E} \times \vec{B}$$

Right-hand screw rule

EM waves carry energy – how much?

e.g., from the Sun to the vinyl seat cover in your parked car….

The energy flow of an electromagnetic wave is described by the **Poynting vector:**

$$\vec{S} \equiv \frac{1}{\mu_0}\vec{E} \times \vec{B}$$

The intensity (I) of a time-harmonic electromagnetic wave whose electric field amplitude is E_0, measured normal to the direction of propagation, is the average over one complete cycle of the wave:

$$I = \frac{P}{A} = S_{\text{avg}} = \frac{1}{2c\mu_0}E_0^2 = \frac{c\epsilon_0}{2}E_0^2$$

WATTS/M^2

P = Power; A = Area; c = speed of light

Key point: intensity is proportional to the *square* of the amplitude of the EM wave

NB. **Intensity = Flux density (F) = Irradiance (*incident*) = Radiant Exitance (*emerging*)**

Electric field of a laser pointer

HE-NEON POWER 1 mWatt, diameter 1 mm^2. How big is the electric field near the aperture (E_0)?

$$I = \frac{P}{A} = S_{\text{avg}} = \frac{1}{2c\mu_0}E_0^2 = \frac{c\epsilon_0}{2}E_0^2$$

A = $\pi r^2 = \pi(5\text{x}10^{-4})^2$ m^2

$$E_0 = \sqrt{\frac{2I}{c\epsilon_0}} = \sqrt{\frac{2(1270\ \text{W/m}^2)}{(3.00 \times 10^8\ \text{m/s})(8.85 \times 10^{-12}\ \text{C}^2/\text{N m}^2)}}$$

$$= 980\ \text{V/m}$$

Radiation Pressure

Radiation also exerts pressure. It's interesting to consider the force of an electromagnetic wave exerted on an object per unit area, which is called the **radiation pressure** p_{rad}. The radiation pressure on an object that absorbs all the light is:

$$F = P/c$$

$$p_{\text{rad}} = \frac{F}{A} = \boxed{\frac{P/A}{c}} = \frac{I}{c}$$

Units: N/m^2

where I is the intensity of the light wave, P is power, and c is the speed of light.

1 Watt m^{-2} = 1 J s^{-1} m^{-2} = 1 N.m s^{-1} m^{-2} = 1 N s^{-1} m^{-1}

Solar sailing

A low-cost way of sending spacecraft to other planets would be to use the radiation pressure on a solar sail. The intensity of the sun's electromagnetic radiation at distances near the earth's orbit is about 1300 W/m^2. What size sail would be needed to accelerate a 10,000 kg spacecraft toward Mars at 0.010 m/s^2?

$$a = F/M = p_{Rad}A/M$$

$$p_{Rad} = I/c$$

$$A = Mac/I$$

$$A = 10^4 \times .01 \times 3 \times 10^8 / 1300 = 23km^2$$

About 4.8 km per side if square

Summary

- Maxwell unified existing laws of electricity and magnetism
- Revealed self-sustaining properties of magnetic and electric fields
- Solution of Maxwell's equations is the three-dimensional wave equation for a wave traveling at the speed of light
- Proved that light is an electromagnetic wave
- EM waves carry energy through empty space and *all remote sensing techniques exploit the modulation of this energy*
- http://www.phy.ntnu.edu.tw/ntnujava/index.php?topic=35